Definitions

- $\text{epi} f := \{(x,\alpha) \mid f(x) \le \alpha\}$ for $(x,\alpha),(x',\alpha') \in X \times \mathbb{R}$, we use $d((x,\alpha),(x',\alpha')) = \max\{\|x-x'\|, |\alpha-\alpha'|\}$
- $\varepsilon\text{-argmin} f := \{x \mid f(x) \le \inf f + \varepsilon\}$
- Truncated Hausdorff distance.

$\hat{d}_\rho(C,D) = \max\{e(C \cap \mathbb{B}(0,\rho); D), e(D \cap \mathbb{B}(0,\rho); C)\}$

where $e(C;D) = \sup_{x\in C} d(x,D) = \sup_{x\in C} \inf_{y \in D} \|x-y\|$.

Prop 2.2 (Royset, 2020)

Let X be a metric space. $(X, \|\cdot\|)$. $f,g : X \mapsto \overline{\mathbb{R}}$ and $\varepsilon, \rho \ge 0$.

$\inf g, \in [-\rho, \rho-\varepsilon)$, $\gamma\text{-argmin} g \cap \mathbb{B}(0,\rho) \ne \phi$, for all $\gamma > 0$.

Then $\inf f - \inf g \le e(\text{epi} g \cap \mathbb{B}(0,\rho); \text{epi} f)$

If in addition, $\inf f, \in [-\rho, \rho-\varepsilon)$, $\gamma\text{-argmin} f \cap \mathbb{B}(0,\rho) \ne \phi$, for all $\gamma > 0$.

$e(\varepsilon\text{-argmin} g \cap \mathbb{B}(0,\rho); \delta\text{-argmin} f) \le e(\text{epi} g \cap \mathbb{B}(0,\rho); \text{epi} f)$

for any $\delta > 2\hat{d}_\rho(\text{epi} f, \text{epi} g) + \varepsilon$.

Proof

Fix $\gamma \in (0, \rho - \varepsilon - \inf g)$. Since $\gamma\text{-argmin} g \cap \mathbb{B}(0,\rho) \ne \phi$,

pick $\bar{x} \in \mathbb{B}(0,\rho)$ s.t $g(\bar{x}) \le \inf g + \gamma < \rho - \varepsilon \le \rho$. and $g(\bar{x}) \ge \inf g \ge -\rho$. Hence $(\bar{x}, g(\bar{x})) \in \text{epi} g \cap \mathbb{B}(0,\rho)$

For any $\gamma > 0$, there exists.

$(x,\alpha) \in \text{epi} f$ s.t $\max\{\|x-\bar{x}\|, |\alpha - g(\bar{x})|\} \le d((\bar{x}, g(\bar{x})), \text{epi} f) + \gamma$.

Then $e(\text{epi} g \cap \mathbb{B}(0,\rho); \text{epi} f) = \sup_{t \in \text{epi} g \cap \mathbb{B}(0,\rho)} d(t, \text{epi} f) \ge d((\bar{x}, g(\bar{x})), \text{epi} f) \ge |\alpha - g(\bar{x})| - \gamma$.

$$\Rightarrow \inf f \le f(x) \le \alpha \le e(\text{epi} g \cap \mathbb{B}(0,\rho); \text{epi} f) + g(\bar{x}) + \gamma$$

$$\le e(\text{epi} g \cap \mathbb{B}(0,\rho); \text{epi} f) + \inf g + 2\gamma \quad (\text{since } \bar{x} \in \gamma\text{-argmin } g)$$

$$\Rightarrow \inf f - \inf g \le e(\text{epi} g \cap \mathbb{B}(0,\rho); \text{epi} f) + 2\gamma \quad \text{for any } \gamma > 0.$$

Next, let $\bar{x} \in \varepsilon\text{-}\operatorname{argmin} g \cap \mathbb{B}(0,\rho)$

then $g(\bar{x}) \le \inf g + \varepsilon < \rho$ and. $g(\bar{x}) \geq \inf g \geq -\rho$. so. $(\bar{x}, g(\bar{x})) \in \operatorname{epi} g \cap \mathbb{B}(0,\rho)$

For any $\gamma > 0$. there exists.

$(x,\alpha) \in \operatorname{epi} f$ s.t $\max\{\|x-\bar{x}\|\ |\alpha - g(\bar{x})|\} \le d((\bar{x}, g(\bar{x})), \operatorname{epi} f) + \gamma. \le e(\operatorname{epi} g \cap \mathbb{B}(0,\rho) ; \operatorname{epi} f) + \gamma$

Then

$$
\begin{aligned}
f(x) \le \alpha &\le g(\bar{x}) + d((\bar{x}, g(\bar{x})), \operatorname{epi} f) + \gamma \\
&\le \inf g + \varepsilon + d((\bar{x}, g(\bar{x})), \operatorname{epi} f) + \gamma \\
&\le \inf f + e(\operatorname{epi} g \cap \mathbb{B}(0,\rho) ; \operatorname{epi} f) \\
&\qquad + \varepsilon + e(\operatorname{epi} f \cap \mathbb{B}(0,\rho) ; \operatorname{epi} g) + 2\gamma \\
&\le \inf f + 2\, \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) + 2\gamma + \varepsilon
\end{aligned}
$$

(This is where assumptions become necessary for both f, g

$\Rightarrow x \in (2\, \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) + 2\gamma + \varepsilon)\text{-}\operatorname{argmin} f$.

$\Rightarrow$ for any $\bar{x} \in \varepsilon\text{-}\operatorname{argmin} g$, there exists $x \in (2\, \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) + 2\gamma + \varepsilon)\text{-}\operatorname{argmin} f$ s.t, $\|x - \bar{x}\| \le e(\operatorname{epi} g \cap \mathbb{B}(0,\rho) ; \operatorname{epi} f) + \gamma$ and $|\alpha - g(\bar{x})| \le e(\operatorname{epi} g \cap \mathbb{B}(0,\rho) ; \operatorname{epi} f) + \gamma$

$\Rightarrow e(\varepsilon\text{-}\operatorname{argmin} f \cap \mathbb{B}(0,\rho) ; (2\eta + \varepsilon + 2\gamma)\text{-}\operatorname{argmin} f) \le e(\operatorname{epi} g \cap \mathbb{B}(0,\rho) ; \operatorname{epi} f) + \gamma$
for any $\gamma > 0$, where $\eta = \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g)$ ∎

This result implies the following;

Thm 4.2 (Royset 2018), Thm 4.1 (Attouch & Wets, 1988)

Let X be a metric space. $(X, \|\cdot\|)$. $f, g : X \mapsto \overline{\mathbb{R}}$ and. $\varepsilon, \rho \geq 0$.

$\inf f, \inf g \in [-\rho, \rho]$, γ-$\operatorname{argmin} f \cap B(0,\rho) \neq \phi$, γ-$\operatorname{argmin} g \cap B(0,\rho) \neq \phi$, for all $\gamma > 0$

$$e(\operatorname{argmin} g \cap B(0,\rho) \,;\, \operatorname{argmin} f) \leq \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) + \psi^{-1}(2\hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g))$$

Proof

$$e(\operatorname{argmin} g \cap B(0,\rho) \,;\, \operatorname{argmin} f)$$
$$\leq e(\varepsilon\text{-}\operatorname{argmin} g \cap B(0,\rho) \,;\, \delta\text{-}\operatorname{argmin} f) + e(\delta\text{-}\operatorname{argmin} f \,;\, \operatorname{argmin} f)$$

For any $x \in \delta$-$\operatorname{argmin} f$, there exists $x_0 \in \operatorname{argmin} f$ s.t.

$$f(x) \leq \inf f + \delta = f(x_0) + \delta \quad \Rightarrow \quad \delta \geq f(x) - \inf f \geq \psi(\operatorname{dist}(x, \operatorname{argmin} f))$$
$$\Rightarrow \text{for any } x \in \delta\text{-}\operatorname{argmin} f, \ \operatorname{dist}(x, \operatorname{argmin} f) \leq \psi^{-1}(\delta)$$
$$\Rightarrow e(\delta\text{-}\operatorname{argmin} f \,;\, \operatorname{argmin} f) \leq \psi^{-1}(\delta)$$

$$e(\operatorname{argmin} g \cap B(0,\rho) \,;\, \operatorname{argmin} f)$$
$$\leq e(\varepsilon\text{-}\operatorname{argmin} g \cap B(0,\rho) \,;\, \delta\text{-}\operatorname{argmin} f) + e(\delta\text{-}\operatorname{argmin} f \,;\, \operatorname{argmin} f)$$
$$\leq e(\operatorname{epi} g \cap B(0,\rho) \,;\, \operatorname{epi} f) + \psi^{-1}(\delta) \quad \text{for } \delta > 2\hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) + \varepsilon. \quad \text{for any } \varepsilon > 0.$$

we have similar stability results for level sets.

$\text{lev}_\varepsilon f := \{ x \in X \mid f(x) \leq \varepsilon \}$

Prep 2.3 (Royset, 2020)

For metric space $(X, \|\cdot\|)$, $f, g : X \to \overline{\mathbb{R}}$, $\rho \in \overline{\mathbb{R}}_+$ and $\delta \in [-\rho, \rho]$

$e(\text{lev}_\delta g \cap \mathbb{B}(0,\rho) ; \text{lev}_\varepsilon f) \leq e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f)$ for $\varepsilon > \delta + e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f)$

Proof

Let $\bar{x} \in \text{lev}_\delta g \cap \mathbb{B}(0,\rho)$ then $g(\bar{x}) \leq \delta \leq \rho$

case 1 : $g(\bar{x}) \geq -\rho$. then $\bar{x} \in \text{epi}\, g \cap \mathbb{B}(0,\rho)$

Fix $\gamma \in (0, \varepsilon - \delta - e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f))$.

then there exists. $(x, \alpha) \in \text{epi}\, f$

$$\max \{ \|x - \bar{x}\|, |g(\bar{x}) - \alpha| \} \leq d((\bar{x}, g(\bar{x})), \text{epi}\, f) + \gamma$$
$$\leq e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f) + \gamma$$

$$\Rightarrow f(x) \leq \alpha \leq g(\bar{x}) + e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f) + \gamma$$
$$\leq \delta + e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f) + \gamma \leq \varepsilon$$

$\Rightarrow x \in \text{lev}_\varepsilon f$ Hence,

$$e(\text{lev}_\delta g \cap \mathbb{B}(0,\rho) ; \text{lev}_\varepsilon f) \leq e(\text{epi}\, g \cap \mathbb{B}(0,\rho) ; \text{epi}\, f) + \gamma$$

case 2 : $g(\bar{x}) < -\rho$. then we can set $g(\bar{x}) = -\rho$

Define $g_\rho(x) = \max \{ g(x), -\rho \}$

then $\text{lev}_\delta g_\rho = \text{lev}_\delta g$ for $\delta \geq -\rho$.

Thus far, perturbation of $\inf f$, δ-$\operatorname{argmin} f$, $\operatorname{argmin} f$ are characterized by epi-distance b/w f and g.

How can we analyze $\hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g)$ for more familiar stats. problem?

Prop 4.1. (Royset, 2020)

Metric space X, $f, g : X \mapsto \bar{\mathbb{R}}$ $\operatorname{epi} f \neq \emptyset$, $\operatorname{epi} g \neq \emptyset$ $\rho \geq 0$.

$$\hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) = \inf \left\{ \eta \geq 0 \;\middle|\; \begin{array}{ll} \inf_{\mathbb{B}(x,\eta)} g(x) \leq \max\{f(x), -\rho\} + \eta & ; \forall x \in \operatorname{lev}_\rho f \cap \mathbb{B}(\rho) \\ \inf_{\mathbb{B}(x,\eta)} f(x) \leq \max\{g(x), -\rho\} + \eta & ; \forall x \in \operatorname{lev}_\rho g \cap \mathbb{B}(\rho) \end{array} \right.$$

Note: The equality used to be $\leq$ unless $(X, \|\cdot\|)$ is finitely compact.

($\leq$) version is old. (Kenmochi, 1974; Theorem 2.1, Attouch & Wets, 1988)

($\geq$) version is, I think, new (Prop 4.1; Royset, 2020)

Proof (lower bound)

Let $\eta = \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g)$. let $(x, f(x)) \in \operatorname{epi} f \cap \mathbb{B}(0,\rho)$

There exists $(\bar{x}, \bar{\alpha}) \in \operatorname{epi} g$ s.t. $\|x - \bar{x}\| \leq \eta + \varepsilon$, $|\bar{\alpha} - f(x)| \leq \eta + \varepsilon$ for any $\varepsilon > 0$.

Then $g(\bar{x}) \leq \bar{\alpha} \leq f(x) + \eta + \varepsilon \leq \max\{f(x), -\rho\} + \eta + \varepsilon$.

Hence $\inf_{\mathbb{B}(x,\eta)} g(x) \leq \max\{f(x), -\rho\} + \eta + \varepsilon$. ; $\forall x \in \operatorname{lev}_\rho f \cap \mathbb{B}(0,\rho)$.

$\Rightarrow \inf\{\eta \geq 0 \mid \inf_{\mathbb{B}(x,\eta+\varepsilon)} g(x) \leq \max\{f(x), -\rho\} + \eta + \varepsilon.\} \leq \hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} g) + \varepsilon$ for any $\varepsilon > 0$

(upper bound)

Let $\inf_{\mathbb{B}(x,\eta)} g(x) \leq \max\{f(x), -\rho\} + \eta$ $\forall x \in \operatorname{lev}_\rho f \cap \mathbb{B}(0,\rho)$

Pick $\bar{x} \in \mathbb{B}(x,\eta)$ s.t

$g(\bar{x}) \leq \max\{f(x), -\rho\} + \eta + \varepsilon$ $\forall x \in \operatorname{lev}_\rho f \cap \mathbb{B}(0,\rho)$

let $\bar{y} = \max\{f(x), -\rho\} - \eta - \varepsilon$ (s.t $\bar{y} \geq g(\bar{x})$ so $(\bar{x}, \bar{y}) \in \operatorname{epi} g$)

Then $\max\{f(x), -\rho\} - \bar{y} = \max\{f(x), -\rho\} - (\max\{f(x), -\rho\} - \eta - \varepsilon) = \eta + \varepsilon$.

$$\Rightarrow \max\{\|x-\bar{x}\|, |\max\{f(x), -\rho\} - \bar{y}|\} \leq \eta+\varepsilon \quad \forall x \in \mathrm{lev}_\rho f \cap \mathbb{B}(0,\rho)$$

Thus

$$\max\{d(x,y), |f_\rho(x) - \bar{y}|\} \leq \eta+\varepsilon \quad \text{where } f_\rho = \max\{f(x), -\rho\}. \ \forall x \in \mathrm{lev}_\rho f \cap \mathbb{B}(0,\rho)$$

$$\Rightarrow \underline{(x, f_\rho(x)) \in (\mathrm{epi}\, g)_{\eta+\varepsilon}} \qquad (A)_\eta := \eta\text{-enlargement of } A$$

(I) $\qquad$ s.t. $\{x \mid t \in A,\ d(x,t) \leq \eta\}$

Note $\hat{d}_\rho(\mathrm{epi} f, \mathrm{epi}\, g) \leq \eta$

$$\Leftrightarrow \mathrm{epi} f \cap \mathbb{B}(0,\rho) \subset (\mathrm{epi}\, g)_\eta \text{ and } \mathrm{epi}\, g \cap \mathbb{B}(0,\rho) \subset (\mathrm{epi} f)_\eta$$

$$\Leftrightarrow (x, f_\rho) \subset (\mathrm{epi}\, g)_\eta \text{ where } f_\rho = \max\{f(x), -\rho\}.$$

$$\forall x \in \mathrm{lev}_\rho f \cap \mathbb{B}(0,\rho)$$

(I) is equivalent to $\hat{d}_\rho(\mathrm{epi} f, \mathrm{epi}\, g) \leq \eta+\varepsilon$.

$$\Rightarrow \hat{d}_\rho(\mathrm{epi} f, \mathrm{epi}\, g) + \varepsilon \leq \inf\{\eta \geq 0 \mid \inf_{\mathbb{B}(x,\eta)} g(x) \leq \max\{f(x), -\rho\} + \eta\} \text{ for any } \varepsilon > 0$$

Kenmochi condition immediately implies the following :

Let $A_\rho = \mathrm{lev}_\rho f \cup \mathrm{lev}_\rho g \cap \mathbb{B}(0,\rho)$

$g(x) \le f(x) + \sup_{A_\rho} |f-g| \quad x \in A_\rho.$

$\Rightarrow \quad \inf_{\mathbb{B}(x,r)} g \le \max\{f(x), -\rho\} + \sup_{A_\rho} |f-g| \quad \forall x \in A_\rho.$ since $x \in \mathbb{B}(x,r)$

Swapping g and f, Kenmochi condition holds w/ $\eta = \sup_{A_\rho} |f-g|$

We have a special result for α-Hölder fn.

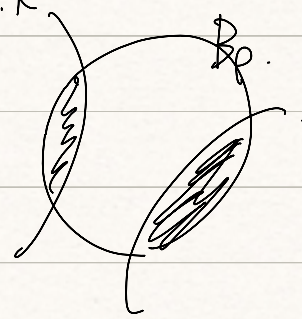
$f : X \mapsto \overline{\mathbb{R}}$ is α-Hölder w/ modulus κ if

$$|f(x) - f(y)| \le \kappa(\rho) \|x-y\|^\alpha \quad \text{for, } x, y \in \mathbb{B}(0,\rho)$$

Prop 4.2 (Royset, 2020)

Let X be a metric space. Suppose f, g are both α-Hölder w/ modulus. κ

Then for any non-empty $C \subset X$,



$$\hat{d}_\rho(\mathrm{epi} f, \mathrm{epi} g) \le \max\{\eta, \kappa(\hat{\rho})\eta^\alpha + \sup_C |f-g|\}$$

where $\hat{\rho} > \rho + \eta$ and $\eta = e(A_\rho; C)$.

Proof Set $\varepsilon \in (0, \hat{\rho} - \rho - \eta)$, and $\eta = e(A_\rho; C)$. Suppose $x \in \mathrm{lev}_\rho X \cap \mathbb{B}(0,\rho)$

There exists. $\bar{x} \in C$ s.t $\|x - \bar{x}\| \le \eta + \varepsilon$. since $x \in \mathrm{lev}_\rho X \cap \mathbb{B}(0,\rho) \subseteq A_\rho$.

$$\inf_{\mathbb{B}(x,\eta+\varepsilon)} g \le g(\bar{x}) \le f(\bar{x}) + \sup_C |f-g|$$
$$\le f(\bar{x}) - f(x) + f(x) + \sup_C |f-g|$$
$$\le \kappa(\eta+\rho)(\eta+\varepsilon)^\alpha + \max\{f(x), -\rho\} + \sup_C |f-g|$$

Swapping the role of f, g,

Kenmochi condition holds w/ $\max\{\eta+\varepsilon, \kappa(\hat{\rho})(\eta+\varepsilon)^\alpha + \sup_C |f-g|\}$ for any $\varepsilon > 0$.

Cor 3.2 (Royset 2020) $\hat{d}_\rho(\mathrm{epi}\, \iota_C, \mathrm{epi}\, \iota_D) \le \hat{d}_\rho(C, D)$

Prop 4.3 (Royset 2020)

Let X be a metric space. $f_i, g_i : X \mapsto \overline{\mathbb{R}}$

where f_1, g_1 are α- Hölder continuous. w/ K.

$\text{epi}(f_1+f_2) \neq \phi$ and $\text{epi}(g_1+g_2) \neq \phi$.

Then $\hat{d}_{\hat{\rho}}(\text{epi}(f_1+f_2), \text{epi}(g_1+g_2)) \leq \sup_{A_\rho}|f_1 - g_1| + \eta + K(\hat{\rho})\eta^{\alpha}$.

where $\eta = \hat{d}_{\bar{\rho}}(\text{epi} f_2, \text{epi}\, g_2)$

$\bar{\rho} \geq \rho + \max\{\sup_{B(0,\rho)}|f_1|, \sup_{B(0,\rho)}|g_1|\}$, $\hat{\rho} \geq \rho + \eta$.

$A_\rho = \text{lev}_\rho(f_1+f_2) \cup \text{lev}_\rho(g_1+g_2) \cap B(0,\rho)$

<u>Proof</u> Let $\varepsilon \in (0, \hat{\rho} - \rho - \eta)$ and $x \in \text{lev}_\rho(f_1+f_2) \cap B_X(\rho)$

$f_2(x) \leq \rho - f_1(x) \leq \bar{\rho}$

① Suppose $f_2(x) \geq -\bar{\rho}$ so $(x, f_2(x)) \in \text{epi} f_2 \cap \mathbb{B}_{X\times\mathbb{R}}(\bar{\rho})$

$\exists (\bar{x}, \bar{\alpha}) \in \text{epi}\, g_2$ w/ $\|x - \bar{x}\| \leq \eta + \varepsilon$, $|f_2(x) - \bar{\alpha}| \leq \eta + \varepsilon$.

$\Rightarrow g_2(\bar{x}) \leq f_2(x) + \eta + \varepsilon$. and.

$$\begin{aligned}\inf_{B_X(x,\eta+\varepsilon)}(g_1+g_2) &\leq g_1(\bar{x}) + g_2(\bar{x}) \\ &= g_1(\bar{x}) - g_1(x) + g_1(x) + g_2(\bar{x}) + f_1(x) - f_1(x) \qquad x \in A_\rho. \\ &\leq K(\hat{\rho})(\eta+\varepsilon)^{\alpha} + \sup_{A_\rho}|f_1 - g_1| + f_2(x) + \eta + \varepsilon + f_1(x) \\ &\leq K(\hat{\rho})(\eta+\varepsilon)^{\alpha} + \sup_{A_\rho}|f_1 - g_1| + \max\{f_1+f_2, -\rho\} + \eta + \varepsilon.\end{aligned}$$

② Next, $f_2(x) < -\bar{\rho}$. then $(x, -\bar{\rho}) \in \text{epi} f_2 \cap \mathbb{B}_{X\times\mathbb{R}}(\bar{\rho})$

$\exists (\bar{x}, \bar{\alpha}) \in \text{epi}\, g_2$ w/ $\|x - \bar{x}\| \leq \eta + \varepsilon$, $|\bar{\alpha} + \bar{\rho}| \leq \eta + \varepsilon$.

$\Rightarrow g_2(\bar{x}) \leq \bar{\alpha} \leq -\bar{\rho} + \eta + \varepsilon$. and.

$$\begin{aligned}\inf_{\mathbb{B}_X(x,\eta+\varepsilon)}(g_1+g_2) &\leq g_1(\bar{x}) + g_2(\bar{x}) \\ &= g_1(\bar{x}) - g_1(x) + g_1(x) - f_1(x) + f_1(x) + g_2(\bar{x}) \\ &\leq K(\hat{\rho})(\eta+\varepsilon)^{\alpha} + \sup_{A_\rho}|g_1 - f_1| + f_1(x) - \bar{\rho} + \eta + \varepsilon.\end{aligned}$$

Since $f_1(x) - \bar{\rho} \leq \sup_{B(\rho)} f_1 - \bar{\rho} \leq -\rho$. (see def of $\bar{\rho}$.)

This establishes.

$$\inf_{B(x,\eta+\varepsilon)} (g_1+g_2) \leq K(\hat{\rho})(\eta+\varepsilon)^{\alpha} + (\eta+\varepsilon) + \sup_{A_\rho}|g_1-f_1| - \rho.$$

Swapping f and g,

Kenmochi condition holds w/

$$K(\hat{\rho})(\eta+\varepsilon)^{\alpha} + (\eta+\varepsilon) + \sup_{A_\rho}|g_1-f_1| \quad \text{w/ any } \varepsilon > 0.$$

Ex). $\xi_1 \cdots \xi_n \overset{iid}{\sim} P$.

$$f(x) = \mathbb{E}\,\psi(\xi,x), \quad f^n(x) = \tfrac{1}{n}\Sigma\,\psi(\xi_i,x)$$

$$\Pr\left\{ \sup_{x\in C} |f(x) - f^n(x)| \geq \delta \right\} \leq \eta(\delta), \quad e(A_\rho; C) \leq \varepsilon$$

$$\hat{d}_\rho(\operatorname{epi} f, \operatorname{epi} f^n) \leq \max\{\varepsilon, K(\hat{\rho})\varepsilon + \delta\} \quad \text{for } \hat{\rho} > \rho+\varepsilon.$$

w/ prob. $1-\eta(\delta)$.

$$f(x) = \mathbb{E}\,\psi(\xi,x) + r(x). \quad f^n(x) = n^{-1}\Sigma\,\psi(\xi_i,x) + r^n(x)$$

ex) $r^n(x) = \sum_{j=1}^{d} s^n(e_j^T x)$.

$$s^n(t) = \begin{cases} \lambda|t| - nt^2/2 & \text{when } |t| \leq \lambda/n \\ \lambda^2/2n & \text{o.w.} \end{cases}$$

ex). $s^n(t) = n^{-1}\sqrt{|t|}$

$$\hat{d}_\rho(\operatorname{epi}(f+r), \operatorname{epi}(f^n+r^n)) \leq \sup_{A_\rho}|r-r^n| + \hat{d}_{\bar{\rho}}(\operatorname{epi} f, \operatorname{epi} f^n)$$
$$+ \mu(\hat{\rho}) \times [\hat{d}_{\bar{\rho}}(\operatorname{epi} f, \operatorname{epi} f^n)]^{\alpha}$$

Ex). Let X be a Hilbert space. $c \in X$.

$\min \langle c, x \rangle$ s.t $x \in K_1$ $\cdots P_1$

$\min . \langle c', x \rangle$. s.t $x \in K_2$ $\cdots P_2$

$K(\cdot) = \max\{\|c\|, \|c'\|\}$.

$\sup_{x \in \mathbb{B}(\rho)} \langle c, x \rangle \leq \rho \cdot \|c\|$. Hence $\bar{\rho} = \rho + \rho\{\|c\| \vee \|c'\|\}$

$$\hat{d}_{\bar{\rho}}(\text{epi } \langle c, x \rangle + i_{K_1}, \text{epi } \langle c', x \rangle + i_{K_2})$$

$$\leq \sup_{A_\rho} |\langle c, x \rangle - \langle c', x \rangle| + \hat{d}_{\bar{\rho}}(i_{K_1}, i_{K_2})$$

$$+ (\|c\| \vee \|c'\|)\, \hat{d}_{\bar{\rho}}(i_{K_1}, i_{K_2})$$

$$\leq \|c - c'\| \times \rho \cdot + (1 + \|c\| \vee \|c'\|)\, \hat{d}_{\bar{\rho}}(K_1, K_2)$$

EX) Minimize $f_0(x)$ sbj to $f_i(x) \le 0$ for $i=1 \cdots m$.

Minimize $g_0(x) + \lambda \sum y_i$ sbj to $g_i(x) \le y_i$, $y_i \ge 0$ for $i=1 \cdots m$

For metric space X, f_0, g_0 are Lipschitz w/ modulus k

$$f(x,y) = \begin{cases} f_0(x) & \text{if } f_i \le 0 \;\; y_i = 0 \text{ for all } i=1 \cdots m \\ \infty & \text{o.w} \end{cases}$$

$$g^{\lambda}(x,y) = \begin{cases} g_0(x) & \text{if } g_i(x) \le y_i,\; y_i \ge 0 \text{ for all } i=1 \cdots m \\ \infty & \text{o.w} \end{cases}$$

Then $\hat{d}_{\rho}(\operatorname{epi} f, \operatorname{epi} g^{\lambda}) \le (1 + k(\hat{\rho}))\max\{ \rho^*/\lambda,\; \psi^{-1}(\rho^*/\lambda)\}$
$+ (1 + m\lambda) \max\limits_{i=0 \cdots m} \sup\limits_{B(\hat{\rho})} |f_i - g_i|$

For $\bar{\rho} > 2\rho + \max\{ \operatorname{dist}((x^{ctr}, 0), \operatorname{epi} f),\; \operatorname{dist}((x^{ctr}, 0), \operatorname{epi} g^{\lambda})\}$

$\rho^* \ge \bar{\rho} + \max\{0,\; -\inf\limits_{B(\bar{\rho})} f_0\}$, $\hat{\rho} \ge \bar{\rho} + \max\{\rho^*/\lambda,\; \psi^{-1}(\rho^*/\lambda)\}$

constraint quantification.

$\max\limits_{i=1 \cdots m} f_i(x) \ge \psi(\operatorname{dist}(x, \operatorname{lev}_0\{\max f_i\}))$ when $x \notin \operatorname{lev}_0\{\max\limits_{i=1 \cdots m} f_i\}$

If $\psi(\gamma) = \gamma^{\beta}$ and $\delta = \max\limits_{i=0 \cdots m} \sup\limits_{B(\hat{\rho})} |f_i - g_i|$ then

$\hat{d}_{\rho}(\operatorname{epi} f, \operatorname{epi} g^{\lambda}) \le C_{\rho} \max\{ \frac{1}{\lambda}, (\frac{1}{\lambda})^{1/\beta}\} + m\lambda\delta$.

pick $\lambda^* = (m\delta)^{\frac{-\beta}{1+\beta}}$

$\le C_{\rho} (m\delta)^{\frac{1}{1+\beta}} \vee C_{\rho} (m\delta)^{\frac{1}{2}}$.

Proof

let $h(x,y) = i_{X\times\{0\}}(x,y) + i_C(x,y)$ $\quad C = \{(x,y) \in X \times \mathbb{R}^m \mid f_i(x) \le y_i,\ y_i \ge 0\}$

$\hat{h}^\lambda(x,y) = \lambda \sum y_i + i_C(x,y)$

$f^\lambda(x,y) = f_0(x) + \hat{h}^\lambda(x,y)$.

$$\hat{d}_\rho(\text{epi} f, \text{epi} g^\lambda) \le \hat{d}_{\bar\rho}(\text{epi}\, g^\lambda, \text{epi} f^\lambda) + \hat{d}_{\bar\rho}(\text{epi} f, \text{epi} f^\lambda) \qquad \bar\rho > 2\rho + \cdots$$

$$\le \underline{\hat{d}_{\bar\rho}(\text{epi}\, g^\lambda, \text{epi} f^\lambda)}\ ① + (1+K(\hat\rho)) \times \underline{\hat{d}_{\rho^*}(\text{epi}\, h, \text{epi}\, \hat{h}^\lambda)}\ ②$$

By the triangle inequality of $\hat{d}_\rho(\cdot,\cdot)$ (prop. 2.1 ; Royset 2020) and Prop 4.3 of Royset 2020.

It remains to bound two epi distances.

② $h(x,y) = i_{X\times\{0\}}(x,y) + i_C(x,y)$

$\hat{h}^\lambda(x,y) = \lambda\sum y_i + i_C(x,y)$ $\quad$ so $h \le \hat{h}^\lambda$ $\quad$ epi $h \supseteq$ epi $\hat{h}^\lambda$

Let $(x,y) \in \text{lev}_{\rho^*} \hat{h}^\lambda \cap \mathbb{B}(0,\rho^*) \in C$

$\lambda \sum y_i \le \rho^*$ and thus $\|y\|_\infty \le \rho^*/\lambda$.

let $\eta = \max\{\rho^*/\lambda,\ \psi^{-1}(\rho^*/\lambda)\}$

① if $f_i(x) \le 0$, $\forall i$

$$\inf_{B((x,y),\eta+\varepsilon)} h \le h(x,0) = 0 \le \max\{\hat{h}^\lambda(x,y), -\rho\}.$$

② if $f_{i^*}(x) > 0$ for some i^*

$$\rho^*/\lambda \ge f_{i^*}(x) \ge \psi(\text{dist}(x, \text{lev}_0\{\max f_i\}))$$

$$\Rightarrow \psi^{-1}(\rho^*/\lambda) \ge \text{dist}(x, \text{lev}_0\{\max f_i\})$$

$\exists \bar{x} \in \mathrm{dist}(x, \mathrm{lev}_0\{\max f_i\})$

s.t. $\quad \|x - \bar{x}\| \le \mathrm{dist}(x, \mathrm{lev}_0\{\max f_i\}) + \varepsilon \le \psi^{-1}(\rho^*/\lambda) + \varepsilon.$

$$\inf_{B((x,y),\eta+\varepsilon)} h \le h(\bar{x},0) = 0 \le \max\{ \hat{h}^\lambda(x,y), -\rho \}.$$

$\Rightarrow$ By Kenmochi condition.

$$\hat{d}_{\rho^*}(h, h^\lambda) \le \max\{ \rho^*/\lambda, \psi^{-1}(\rho^*/\lambda) \}.$$

① . $\hat{h}^\lambda(x,y) = \lambda \Sigma y_i + i_C(x,y)$

$C = \{(x,y) \in X \times \mathbb{R}^m \mid f_i(x) \le y_i, \ y_i \ge 0 \}$

$f^\lambda(x,y) = f_0(x) + \hat{h}^\lambda(x,y)$, $\quad g^\lambda(x,y) = g_0(x) + \hat{h}^\lambda(x,y)$.

Let $\delta = \max_{0 \dots m} \sup_{\mathbb{B}(0,\bar{\rho})} |f_i - g_i|$ and $(x,y) \in \mathrm{lev}_{\bar{\rho}} f^\lambda \cap \mathbb{B}(0,\bar{\rho})$

Then $(x,y) \in C$, $f_i(x) \le y_i$ and $g_i(x) \le y_i + \delta$ for all $i \in \{0, \cdots m\}$

Set $\eta = (1+m\lambda)\delta$ and $\bar{y} = y + (\delta, \cdots, \delta)^T$

$$\begin{aligned} \inf_{B((x,y),\eta)} g^\lambda \le g^\lambda(x,\bar{y}) &= g_0(x) + \lambda \Sigma \bar{y}_i \\ &\le f_0(x) + \delta + \lambda \Sigma y_i + \lambda m \delta. \\ &= f_0(x) + \lambda \Sigma y_i + \delta(1+\lambda m) = f^\lambda + \eta \end{aligned}$$

$\Rightarrow$ By Kenmochi condition. $\hat{d}_{\rho^*}(g^\lambda, f^\lambda) \le (1+m\lambda) \max_{0 \dots m} \sup_{\mathbb{B}(0,\bar{\rho})} |f_i - g_i|$
